Ohio Council of Teachers of Mathematics
Solutions to Thirty-Fourth Annual Contest
February 24, 2007

1) 2005 — $2007 - 2 - 0 - 0 \cdot 7 = 2007 - 2 = 2005$

2) A) F — 2007 is not a perfect square since $\sqrt{2007}$ is not an integer

B) F — $2007 = 3^2 \cdot 223$, and thus is not prime.

C) T — $2007 = \frac{2007}{1}$, and thus is a rational number.

3) side

4) 2 yd 1 ft 9 in — Note 4 yds, 1ft, 4 in. = 3yds, 4 ft, 4 in. = 3 yds, 3 ft, 16 in. Thus, (4yds, 1ft, 4 in.) minus (1 yd, 2 ft, 7 in) = (3yds, 3ft, 16 in.) minus (1 yd, 2 ft, 7 in) = (2 yd, 1 ft, 9 in).

5) AAA — Triangles with 3 congruent angles are similar triangles, and not necessarily congruent.

6) 55% — Net income = $\$500 - 0.25 \cdot \$500 - \$100 = \$500 - \$125 - \$100 = \$275$. Thus, his net income is $\frac{275}{500} \cdot 100\% = 55\%$ of his gross income.

7) 876593 — The largest six digit number with distinct digits is 987654. However, we must move the 9 to the ten's place, so the new largest number would be 876594. But we must have an odd number, so we get 876593.

8) C — $\frac{32^{4004}}{64^{4004}} = \left(\frac{1}{2}\right)^{4004}$, so not A. $\frac{64^{2002}}{64^{4004}} = \frac{1}{64^{2002}}$, so not B.

$64^{4004} = 2^{6 \cdot 4004} = 2^{24024}$, so $\frac{2^{24023}}{64^{4004}} = \frac{2^{24023}}{2^{24024}} = \frac{1}{2}$ so YES for C.

$\frac{64^{4003}}{64^{4004}} = \frac{1}{64}$, so not D. $\frac{4^{12011}}{64^{4004}} = \frac{4^{12011}}{4^{3 \cdot 4004}} = \frac{4^{12011}}{4^{12012}} = \frac{1}{4}$, so not E.

9) -130 — $200x + 7 = 2007 \Rightarrow 200x = 2000 \Rightarrow x = 10$. Thus, $7x - 200 = 70 - 200 = -130$.

10) 1 — $\frac{200}{7} = 28.\overline{571428}$, so the decimal has 6 digits that repeat. $6\overline{)2007}$ with quotient $334R3$, so we need the 3rd repeating digit after the decimal.

11) $\frac{a}{b} = \frac{17}{20}$ — As long as the sum of the lengths of the two smaller sticks is larger than the length of the 3rd stick, then the sticks can form a triangle. The number of ways to select three sticks is $\binom{6}{3} = 20$. The only selections that will not form a triangle are (3,4,7), (3,4,8), and (3,5,8). Thus, 17 of the 20 selections will form a triangle and the probability is $\frac{17}{20}$.

12) $b = -1, c = -12, d = 0$ — Since $a = 1$, we know that $f(x) = (x+3) \cdot x \cdot (x-4) = x^3 - x^2 - 12x + 0$.

13) $-\frac{4\sqrt{7}}{7}$ or $-\frac{4}{\sqrt{7}}$ $\sin^2(\theta)+\cos^2(\theta)=1 \Rightarrow \sin^2(\theta)=1-\left(\frac{3}{4}\right)^2=\frac{7}{16} \Rightarrow \sin(\theta)=\pm\frac{\sqrt{7}}{4}$. Since θ is in the 4th quadrant, sin(θ) is negative. Thus, $\csc(\theta)=\frac{1}{\sin(\theta)}=-\frac{4}{\sqrt{7}}=-\frac{4\sqrt{7}}{7}$.

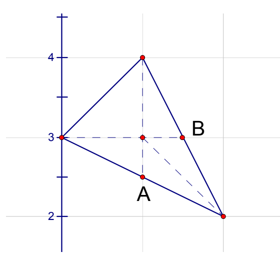
14) A A regular n-gon has n lines of symmetry. If n is odd, each line of symmetry passes through a vertex and the midpoint of the opposite side. If n is even, each line of symmetry either connects opposite vertices or midpoints of opposite sides.

15) XVII MDCLXVI / XCVIII = 1666/98 = 17 = XVII

16) $3+2\sqrt{3}$ The third angle is $180°-60°-45°=75°$. Recall $\tan(60°)=\sqrt{3}$ and $\tan(45°)=1$. Thus, $3+2\sqrt{3}=\sqrt{3}+1+\tan(75°) \Rightarrow \tan(75°)=\sqrt{3}+2$. Thus, the product of the tangents is $\sqrt{3}\cdot 1\cdot\left(\sqrt{3}+2\right)=3+2\sqrt{3}$.

17) 10200_{three} Note $8_{ten}=22_{three}$. Thus, $10101_{three}+22_{three}=10200_{three}$.

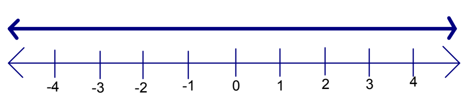
18) 1.5 or $\frac{3}{2}$ sq. units. Divide the triangle with the dashed lines displayed in the picture.



Note that the displayed grid consists of squares with area 1, and points A and B are the midpoints of the sides of one of these squares. Thus, the four "smaller" triangles each have a height of 1 and base of $\frac{1}{2}$ and thus each have area $\frac{1}{4}$. Clearly the larger triangle has area $\frac{1}{2}$. Thus, the total area is $4\cdot\frac{1}{4}+\frac{1}{2}=\frac{3}{2}$.

19) 40 Matt has received a total of $5\cdot 88=440$ points. Note it is possible for Matt to receive 100's on 4 of the exams and have this point total. Thus, his lowest possible score on an exam is 440 – 400 = 40.

20)



The solution set of $\{x : x^2 \le 4\}$ is the interval $[-2,2]$. The solution set of $\{x : x^2 > 1\}$ is $(-\infty,-1)\cup(1,\infty)$. Thus, the union of these two solution sets is the entire number line.

21) 29.5% The effective raise is $(1.10\cdot 1.09\cdot 1.08-1)\cdot 100\% = 29.492\%$.

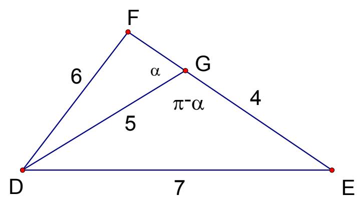
22) $\frac{24}{7}$ or $3\frac{3}{7}$ or 3.4286 The angle bisector theorem states that the angle bisector of a triangle divides the opposite sides into the same ratio as the sides adjacent to the angle. Thus, $\frac{x}{4}=\frac{6}{7} \Rightarrow x=\frac{24}{7}$. This problem can also be solved using the law of sines.

23) $\frac{1}{2}$ or 0.5 The 4th through 7th terms are 4, 2, 1, $\frac{1}{2}$.

24) D and F A is the distributive property. B follows from the commutative property for multiplication. C is the associative property for multiplication. D follows from the commutative property for addition. E is the associative property for addition. F follows from the commutative property for addition.

25) $\frac{a}{b}=\frac{12}{17}$ Let F be the number of females and M be the number of males in Mathburg. Then $\frac{3}{4}F=\frac{2}{3}M \Rightarrow M=\frac{9}{8}F$. The fraction of adults married is $\dfrac{\frac{3}{4}F+\frac{2}{3}M}{F+M}=\dfrac{\frac{6}{4}F}{\frac{17}{8}F}=\dfrac{12}{17}$.

26) $1+2\sqrt{3}$ or $1+\sqrt{12}$ or 4.4641 Let α be the angle as labeled in the picture. Then using the law of cosines,



$49=25+16-20\cos(\pi-\alpha)\Rightarrow 8=20\cos(\alpha)$ since $\cos(\pi-\alpha)=-\cos(\alpha)$. Thus, $\cos(\alpha)=\frac{2}{5}$. Applying the law of cosines again, we get $36=25+y^2-5y\cos(\alpha)=25+y^2-2y$. Thus, $y^2-2y-11=0$. Using the Quadratic Formula, we have $y=\dfrac{2\pm\sqrt{4+44}}{2}=\dfrac{2\pm4\sqrt{3}}{2}=1\pm2\sqrt{3}$, but y must be positive, so $y=1+2\sqrt{3}$.

27) 5040 or $\dfrac{8!}{2!\cdot2!\cdot2!}$ or 7! There are 8 letters in cassette, and the number of permutations of 8 distinct objects is 8!. However, there are 2 s's. 2 e's, and 2 t's, so we divide by the number of ways (2!) to permute each of these letters (if they were treated as distinct).

28) $\frac{7}{3}$ or $2\frac{1}{3}$ or 2.3333 $\log_3 27-\log_4 4+\log_{27}3-\log_{25}1=3-1+\frac{1}{3}-0=\frac{7}{3}$.

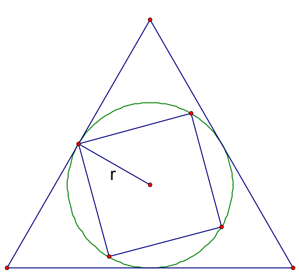
29) E The value of the mean cannot be determined since the box and whiskers plot uses the median, not the mean.

30) $\frac{45}{2}$ or $22\frac{1}{2}$ or 22.5 $y=k\dfrac{x}{z}$. Thus, $60=k\dfrac{40}{10}\Rightarrow k=15$. Thus, $y=15\cdot\dfrac{45}{30}=\dfrac{45}{2}$ when $z=30$ and $x=45$.

31) 10 The 3rd row needs a 2, 6, and 3 in it. Thus, G must be 6 since its 3x3 box already has a 2 and a 3 in it. Then H must be 3 and y must be 2. The middle 3x3 box would still need a 4 and an 8, so since the top row already has a 4, D must be 4 and A must be 8. The 1st 3x3 box still needs a 5 and 8, and since A is 8, we must have C is 8 and x is 5. Thus, $xy=10$.

4	x(5)	6	9	3	A (8)	2	B(7)	1
C(8)	3	1	D(4)	7	2	E	F	6
9	y(2)	7	1	5	G(6)	H(3)	4	8

32) 24 sq. units Since the side length of the triangle is 12, the length of the altitude of the triangle is $\dfrac{\sqrt{3}}{2}\cdot12=6\sqrt{3}$. Thus the radius of the circle is $r=\dfrac{6\sqrt{3}}{3}=2\sqrt{3}$. If x is the length of the side of the square, then applying the Pythagorean Theorem to one of the small triangles created by the diagonals of the square, we get $x^2=2r^2=2(12)=24$.



33) \$12.10 per shirt $\frac{3\cdot\$11+3\cdot\$12+4\cdot\$13}{10\text{ shirts}}=\frac{\$121}{10\text{ shirts}}=\$12.10/\text{shirt}$.

34) 1053

$$(-2i^2)(2007)+\sum_{k=1}^{2}207k-(2\sqrt{6})^2(27)+n=7!$$
$$4014+207+414-648+n=5040$$
$$n=5040-3987=1053$$

35) 3 hr. 59 min. The three teachers work at rates of $\frac{1}{11},\frac{1}{12},\frac{1}{13}$ sets of tests per hour individually. Thus, they work at a rate of $\frac{1}{11}+\frac{1}{12}+\frac{1}{13}=\frac{431}{1716}$ sets of tests per hour when working together. Thus, it would take them $\frac{1716}{431}\approx 3.98144$ hours to grade one set of like tests together. $.98144\cdot 60\text{ min.}\approx 58.89\text{ min}$.

36) $\frac{4\sqrt{5}}{5}$ or $\frac{4}{\sqrt{5}}$ or 1.7889 The line perpendicular to $y=0.5x+4$ at its y-intercept is $y=-2x+4$. This line intersects the line $y=0.5x+2$ at the point $\left(\frac{4}{5},\frac{12}{5}\right)$. Thus, the distance between the lines is $\sqrt{\left(\frac{4}{5}-0\right)^2+\left(\frac{12}{5}-4\right)^2}=\sqrt{\frac{16}{25}+\frac{64}{25}}=\sqrt{\frac{80}{25}}=\frac{4\sqrt{5}}{5}$.

37) $\sqrt{148}=2\sqrt{37}\approx 12.1655$ inches. Let s be the length of an edge of the cube. The surface area of the rectangular solid is $2(16\cdot 12+16\cdot 9+12\cdot 9)=888=6s^2$, so $s=\sqrt{148}$.

38) slide rule

39) $\sqrt{29}\approx 5.3852$ Note this distance is the same as the corresponding distance for the congruent ellipse centered at the origin, namely the ellipse given by $\frac{x^2}{4}+\frac{y^2}{25}=1$. But this is simply the distance between the points (2,0) and (0,5), which is $\sqrt{4+25}=\sqrt{29}$.

40) 126 mph Let p be the average air speed of the plane and w be the wind speed. Then, $2(p+w)=280\Rightarrow p+w=140$ and $2.5(p-w)=280\Rightarrow p-w=112$. Adding these equations, we get $2p=252\Rightarrow p=126$.

The names of the students who qualify for the OHMIO competition will be notified by email by Sunday, March 4
Your score will be posted on your home page at the OCTM Tournament website (www.octmtournament.org/)

Solutions provided by: **Dr. Christopher Swanson, Ashland University**
Dr. Gordon Swain, Ashland University