Ohio Council of Teachers of Mathematics
Solutions to Thirty-Third Annual Contest
February 25, 2006

1) 2 — $2 + (0 \cdot 0) \cdot 6 = 2 + 0 = 2$

2) 2.006×10^3

3) rhombus

4) A — Her speed is constant initially, and then increases as she goes downhill.

5) $k = 7$ — The given line is horizontal, so the parallel line has equation $y = 7$ and thus the *y*-coordinate of (2006,*k*) must be 7.

6) 59 — $2006 = 2 \cdot 17 \cdot 59$

7) 89 — The smallest possible value for *a* is 3, so the smallest possible value for 2*a* is 6. Then the smallest possible value for *b* is 7, and 3*b* is at least 21, *c* is at least 22, 4*c* is at least 88, and finally *d* is at least 89.

8) $\frac{7}{12}$ — The distance between U and Z is $\frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$. The (U,Z) interval is divided in to 5 congruent subintervals, each of length $\frac{1}{12}$. Thus, $Y = \frac{2}{3} - \frac{1}{12} = \frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

9) 45 points/game — The total number of points scored in the 21 games is $56 \cdot 21 = 1176$. The total number of points scored in the11 home games is $66 \cdot 11 = 726$. Thus the total number of points scored in the 10 away games is $1176 - 726 = 450$, and hence they averaged $\frac{450}{10} = 45$ points/away game.

10) 300 votes — Let *K*, *J*, and *L* be the number of votes received by Kenneth, Judy, and Linda. Then $K = \frac{5}{6}J$ and $J = \frac{4}{5}L$. Hence, $L = \frac{5}{4}J = \frac{5}{4}(\frac{6}{5}K) = \frac{3}{2}K = 300$.

11) $\frac{1}{4} = 0.25$ — The probability of getting a tail is 0.5. The probability of getting a prime number roll {2,3,5} is 0.5. Since these events are independent of each other , the probability of both events occurring is (0.5)(0.5)=0.25.

12) A —
A) $x^3 = x \Leftrightarrow x^3 - x = 0 \Leftrightarrow x(x-1)(x+1) = 0$, so 3 unique real solutions.
B) $x^3 = -x \Leftrightarrow x^3 + x = 0 \Leftrightarrow x(x^2+1) = 0$, so 1 real solution and 2 complex solutions.
C) $x^3 + 4x^2 + 4x = 0 \Leftrightarrow x(x+2)^2 = 0$, so 2 real solutions
D) $x^3 - x^2 - x + 1 = 0 \Leftrightarrow x^2(x-1) - (x-1) = 0 \Leftrightarrow (x^2-1)(x-1) = 0$
$\Leftrightarrow (x+1)(x-1)^2 = 0$, so 2 real solutions.

13) Lowe: 30, Reinhardt: 30 — Let *L* and *R* be the number of tests Lowe and Reinhardt had originally. Then, $L - 6 = \frac{2}{3}(R+6) \Rightarrow L = \frac{2}{3}R + 10$ and $R - 10 = \frac{1}{2}(L+10) \Rightarrow R = \frac{1}{2}L + 15$. Hence, $L = \frac{2}{3}(\frac{1}{2}L + 15) + 10 \Leftrightarrow \frac{2}{3}L = 20 \Leftrightarrow L = 30 \Rightarrow R = 30$.

14) $x = -6$ — $200(6 + x) = 0 \Leftrightarrow 6 + x = 0 \Leftrightarrow x = -6$

15) $e = -\frac{4}{3} = -1.\overline{3}$ — Multiplying the matrices, we get $3y = d + 1 = -6$, so $y = -2$. Then $2y = 3e \Leftrightarrow e = \frac{2}{3}y = \frac{2}{3}(-2) = -\frac{4}{3}$.

16) 162 pounds — Let b be Batman's weight in pounds. To balance, we must have $180 \cdot 9 = 10b \Leftrightarrow b = 162$

17) $f(0) = -1$ — $f(0) = 0^3 - 1 = -1$

$f(-2) = 4$ — $f(-2) = |-2-2| = |-4| = 4$

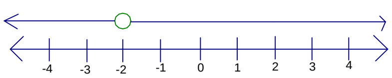
$f(3) = 26$ — $f(3) = 3^3 - 1 = 27 - 1 = 26$

18) $x = -\frac{31}{10} = -3.1$ — $\left(\frac{1}{4}\right)^{2x+8} = 8^{2x+5} \Leftrightarrow \left(2^{-2}\right)^{2x+8} = \left(2^3\right)^{2x+5} \Leftrightarrow 2^{-4x-16} = 2^{6x+15}$

$\Leftrightarrow -4x - 16 = 6x + 15 \Leftrightarrow -10x = 31 \Leftrightarrow x = -\frac{31}{10}$

19) 1 — $x^2 - 11x + 24 = (x-8)(x-3)$, and so the sum of these linear factors is $2x - 11$. $x^2 - 10 - 24 = (x-12)(x+2)$, and so the sum of these linear factors is $2x - 10$. Thus, the answer is $2x - 10 - (2x - 11) = 1$. Note for calculus students: In general, the sum of the linear factors of $f(x) = x^2 + bx + c$ is $f'(x)$. Show this is true.

20)



$x^2 - x - 6 < 0 \Leftrightarrow (x-3)(x+2) < 0 \Leftrightarrow -2 < x < 3$

$x^2 + x - 2 > 0 \Leftrightarrow (x+2)(x-1) > 0 \Leftrightarrow x < -2 \text{ or } x > 1$

The only value of x that does not satisfy at least one of these final inequalities is –2.

21) 1969 — $\frac{12.79}{1 + 2.402e^{-0.0309t}} = 10 \Leftrightarrow 12.79 = 10(1 + 2.402e^{-0.0309t}) \Leftrightarrow \frac{0.279}{2.402} = e^{-0.0309t}$

$\Leftrightarrow t = \frac{1}{-0.0309}\ln\left(\frac{0.279}{2.402}\right) \approx 69.7$. Alternatively, graph $P(t)$ and the horizontal line $y = 10$ in your calculator and find where they intersect.

22) $w = 5i + 3j$ — $w = v + u = (3i + 4j) + (2i - j) = 5i + 3j$

23) $m = -\frac{1}{2}$ $b = \frac{5}{2}$ — Note $t = y - 2$, so $x = 1 - 2(y-2) = -2y + 5 \Rightarrow y = -\frac{1}{2}x + \frac{5}{2}$.

24) no solution OR { } OR ∅ OR impossible — Given any real number k, $|x - k| = 2$ has two solutions, namely $x = k + 2$ or $x = k - 2$, the two real numbers exactly 2 units away from k.

25) $a = 1, b = -1$ — Note $i^{2006} = i^{2004} \cdot i^2 = 1 \cdot (-1) = -1$. Thus,

$$\frac{2006 + i^{2006} - i^{-2006} - 2006i}{2006} = \frac{2006 - 1 - (-1)^{-1} - 2006i}{2006} = 1 - i$$

26) 27 cubic inches — A cube has 12 edges of equal length, and thus the largest cube made from 36 inches of wire has 3 inches of wire on each edge and has volume $3^3 = 27$ cubic inches.

27) $mPO = 129°$ — Note $m\angle MQN = 86°$. Thus, $mMN + mPO = 2 \cdot 86$. Hence, $4mMN = 172 \Leftrightarrow mMN = 43 \Rightarrow mPO = 3 \cdot 43 = 129$.

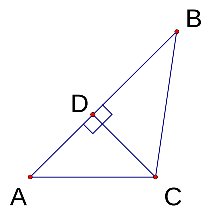
28) 2 and 3 — $\log_{20.06} 2006 = \frac{\ln 2006}{\ln 20.06} \approx 2.54$. Alternatively, note $20.06^2 \approx 20^2 = 400$ and $20.06^3 \approx 20^3 = 8000$.

29) 240 ft. Let the pool have sides of length x and $2x$. Then the border has area $2800 = 2 \cdot 10x + 2 \cdot 20x + 4 \cdot 100 = 60x + 400 \Rightarrow 60x = 2400 \Rightarrow x = 40$. Thus, the perimeter is $6x = 240$ feet.

30) 0 $x^3 - 8 = (x - r_1)(x - r_2)(x - r_3)$ where r_1, r_2, r_3 are the three cube roots of 8. If we expand the 2nd expression, the coefficient of x^2 is $-(r_1 + r_2 + r_3)$, which must equal 0. Thus, the sum of the roots is 0.

31) $(2, -\frac{39}{8})$ or $(2, -4\frac{7}{8})$ or $(2, -4.875)$ Completing the square, $y = 2(x-2)^2 - 5$. Thus, the vertex of the parabola is at (2,-5). The latus rectum is $\frac{1}{2}$ and thus the distance from the vertex to the focus is $\frac{1}{8}$. The parabolas opens up, and thus we add $\frac{1}{8}$ to the y-coordinate of the vertex.

32) $14\sqrt{2} = \sqrt{392} \approx 19.7990$ Consider the following picture of ΔABC, where $\overline{CD}$ is the altitude drawn from C to side $\overline{AB}$. (Note: We know angles A and B are both acute angles based on the values of their tangents, and thus the altitude is in the interior of the triangle.)



$m\angle A = 45°$, so $AD = DC$ and by the Pythagorean Theorem, $AD^2 + DC^2 = 144 \Rightarrow 2AD^2 = 144 \Rightarrow AD = \sqrt{72} = 6\sqrt{2}$. Next note $\frac{3}{4} = \tan B = \frac{DC}{DB} = \frac{6\sqrt{2}}{DB}$, so $DB = 8\sqrt{2}$. Thus, $c = 6\sqrt{2} + 8\sqrt{2} = 14\sqrt{2}$.
Note: the approximate answer can be found by using a calculator to approximate the measure of angles B and C, and then using the law of sines.

33) $\sum_{k=1}^{6} k^2$

34) 14.4 km/hr $\frac{800\text{m}}{\frac{10}{3}\text{min.}} \cdot \frac{1\text{km}}{1000\text{m}} \cdot \frac{60\,\text{min.}}{1\text{hr.}} = \frac{144}{10}\text{km/hr} = 14.4\text{km/hr}$.

35) 24 cards Warmbrodt received 10 cards, so 8 cards was one half of Wohlever’s cards. Wohlever received 16 cards, so 14 cards was one half of Holloway’s cards. Holloway received 28 cards, so 26 was half of Hower’s cards. Hower received 52 cards, but gave 28 cards to Holloway and thus still had 24 cards at the end.

36) $6\sqrt{6} = \frac{18\sqrt{2}}{\sqrt{3}} \approx 14.6969$ cm. Note if the trapezoid has area equal to one third of the area of the large triangle, the triangle formed by the segment has an area equal to two-thirds of the area of the large triangle. These two triangles are similar, so the ratio of the sides must be $\sqrt{\frac{2}{3}}$. Hence, the segment must have length $18\sqrt{\frac{2}{3}} = 6\sqrt{6}$.

37) Brahmagupta or 30 square units Brahmagupta was an Indian mathematician and astronomer. He wrote the first text to consider 0 as a number. Brahmagupta's formula for the area of an circumscribed quadrilateral with side lengths *a*, *b*, *c*, and *d* is $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where *s* is the semi-perimeter, i.e. $s = \frac{a+b+c+d}{2}$. In particular, Heron's formula for the area of a triangle can be derived from this by considering one side of the quadrilateral to have length 0. For the given quadrilateral, the semiperimeter is 12, and thus the area is $\sqrt{(12-9)(12-6)(12-2)(12-7)} = \sqrt{3 \cdot 6 \cdot 10 \cdot 5} = 30$.

38) $\sqrt{\frac{1849}{136}} \approx 3.6872$ First, we find the equation of the line that passes through (2,-3) and is perpendicular to the given line. The slope of this line is $\frac{5}{3}$, and thus $y + 3 = \frac{5}{3}(x-2) \Leftrightarrow y = \frac{5}{3}x - \frac{9}{3}$. The two lines intersect at $\left(\frac{265}{68}, \frac{11}{68}\right)$. The distance between this point and (2,-3) is $\sqrt{\left(2-\frac{265}{68}\right)^2 + \left(-3-\frac{11}{68}\right)^2} = \sqrt{\frac{1849}{136}}$.

39) 9 marbles If do not have 3 marbles of any one color, then we have at most 2 marbles of each color, for a total of 8 marbles. Thus, as soon as we draw a 9th marble, we are guaranteed 3 marbles are of the same color. This is an example of a famous mathematical result known as the (generalized) Pigeonhole Principle.

40) 60.3mph The Browns bus must do the entire trip in 6.25 hours. The fans stop to eat for 1.5 hours and traveled initially for $\frac{57.8}{150}$ hours. Thus, the bus must travel the remaining 130 miles in $6.25 - 1.5 - \frac{57.8}{150}$ hours. Hence, the bus must travel at $\frac{130}{6.25 - 1.5 - \frac{57.8}{150}} \approx 60.3$ mph.

The names of the students who qualify for the OHMIO competition will be posted on the OCTM Tournament website (www.octmtournament.org/) by noon on Monday, March 6.

Solutions provided by: Dr. Christopher Swanson, Ashland University