Ohio Council of Teachers of Mathematics
Solutions to Thirtieth Annual Contest
February 22, 2003

1) 1 $\left(2^0 + 0\right)^3 = (1+0)^3 = 1^3 = 1$

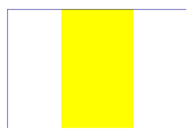
2) 1936 $\sqrt{2003} = 44.754...$, so largest square less than 2003 is $44^2 = 1936$

3) Impossible $|x| + 3 = 2$? $|x| = -1$, but $|x|$? 0.

4) A terminating decimal $3.1 = \frac{31}{100}$, repeating decimal $3.\overline{1} = \frac{28}{9}$, $-\frac{2}{13}$, 0 are all rational

5) $x = 9$ $200x + 3x = 2003 + 3x - 203$? $200x = 1800$? $x = 9$

6) 28 square units



$AD = 10$ and $AC = 7$, so $CD = 3$
$BD = 7$ and $CD = 3$, so $BC = 4$
Thus, the shaded region is a 4 unit by 7 unit rectangle and thus has area $4 ? 7 = 28$ square units.

7) 2 $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{6+3+2+1}{6} = \frac{12}{6} = 2$

8) L J demonstrates multiplication of 2 and $(7+3)$ is commutative.
K demonstrates addition of 7 and 3 is commutative.
L demonstrates multiplication of *c*, *d*, and $(a+b)$ is associative.
M demonstrates multiplication of *c* and *d* is commutative.

9) T The repeating pattern QWERTY has 6 letters. $\frac{2003}{6} = 333.833...$, so the remainder of $\frac{2003}{6}$ is $2003 - 6 ? 333 = 5$, and T is the 5th letter in the pattern.

10) 49 The upper, left corner of a 2 x 2 square can be placed on any of the squares in the upper, left corner of the 7 x 7 square to form a 2 x 2 square on the checkerboard. Thus, $7 ? 7 = 49$ squares are possible.

11) 1903 The greatest number that can be used will be the one matched with the 10 smallest numbers possible. The 10 smallest numbers possible are 1, 3, 5, 7, 9, 11, 13, 15, 17, and 19 are an arithmetic sequence with sum $\frac{(1+19)?10}{2} = 100$. Thus, the greatest number is $2003 - 100 = 1903$.

12) 2012 $\frac{x_1+x_2+x_3+x_4}{4} = 2003$ and $\frac{x_1+x_2+x_3}{3} = 2000$? $x_1 + x_2 + x_3 = 6000$. Thus, $\frac{6000+x_4}{4} = 2003$? $6000 + x_4 = 8012$? $x_4 = 2012$.

13) 13, 14 If the base has length 4, then there are two sides of length 5 and the perimeter is $4+5+5 = 14$. If the base has length 5, then there are two sides of length 4 and the perimeter is $5+4+4 = 14$. (Note in either case the triangle exists since the sum of the lengths of any two sides is greater than the length of the third side.)

14) $\frac{16}{625}$ The prime numbers between 0 and 9 are 2, 3, 5, and 7. Thus, the probability one digit is prime is $\frac{4}{10} = \frac{2}{5}$ and the probability all four are prime is $\frac{2}{5} ? \frac{2}{5} ? \frac{2}{5} ? \frac{2}{5} = \frac{16}{625}$.

15) 40 Noting that 2003 is prime, the positive integral factors of $2^3 \leftrightarrow 3^4 \leftrightarrow 2003^1$ are integers of the form $2^r \leftrightarrow 3^s \leftrightarrow 2003^t$ with $0 \le r \le 3$, $0 \le s \le 4$, and $0 \le t \le 1$. Thus, there are 4 possible values for *r*, 5 possible values for *s* and 2 possible values for *t*, and the number of integral factors is $4 ? 5 ? 2 = 40$.

16) $\frac{2003h}{t}$ miles The car's speed is $\frac{2003}{t}$ mph, and thus in *h* hours it travels a distance of $\frac{2003h}{t}$ miles.

17) 2mn Dropping down altitudes to the longer base, we know that $AD = 2m + n$ and $BC = 2m - n$ so

$AB + CD = 2m + n - (2m - n) = 2n$ with $AB = CD$ which gives $AB = n$. Since the base angles measure 45°, the two small triangles are 45°-45°-90° triangles and thus, the height of the trapezoid is also *n*, giving its area as $\frac{(2m + n + 2m - n)n}{2} = 2mn$.

18) 6, 8, 10 The smallest Pythagorean triple is 3, 4, 5. To make this composite, multiply each of these numbers by 2. This will be the smallest with composite numbers since the next smallest Pythagorean triple is 5, 12, 13.

19) a) + b) + c) 0 a) is clearly true. b) is true as can be seen by adding *w* to the given inequality $x > y$. c) is always false as can be seen by starting with the inequality $y > z$, multiplying by –1 and flipping the sign to give $-y < -z$, and adding *w* to give $w - y < w - z$.

20) Note the two absolute values equal zero when *x* is $-\frac{3}{2}$ or 6. Thus, $2x + 3 + x - 6 > 9 ? \; 3x - 3 > 9 ? \; x > 4$ which is true. If $-\frac{3}{2} < x < 6$, the inequality becomes $2x + 3 - x + 6 > 9 ? \; x + 9 > 9 ? \; x > 0$. If $x \le -\frac{3}{2}$, the inequality becomes $-2x - 3 - x + 6 > 9 ? \; -3x + 3 > 9 ? \; x < -2$. Combining these results gives the answer.

21) \$1109 If the initial investment is *P*, then $P ? (1.03)^{20} = 2003 ? \; P = \frac{2003}{(1.03)^{20}} \quad 1109.01$

22) 31.8% Price for a candy bar yesterday was \$1.10, while the price of a candy bar today is \$0.75. Today's price is $\frac{.75}{1.1} * 100\% = 68.18\%$ of yesterday's price, so the percent decrease is $100\% - 68.18\% = 31.82\%$.

23) $\cot\theta$ $\frac{\cot\theta - 1}{1 - \tan\theta} = \frac{\cot\theta - 1}{1 - \tan\theta} ? \frac{\cot\theta}{\cot\theta} = \frac{(\cot\theta - 1)\cot\theta}{\cot\theta - 1} = \cot\theta$.

24) $(-1, 3)$ Any line to $y = 2x$ will have slope $-\frac{1}{2}$. Thus, the line through the point $(3,1)$ to $y = 2x$ is given by $y - 1 = -\frac{1}{2}(x - 3) ? \; y = -\frac{1}{2}x + \frac{5}{2}$. This line intersects $y = 2x$ when $2x = -\frac{1}{2}x + \frac{5}{2} ? \; \frac{5}{2}x = \frac{5}{2} ? \; x = 1, y = 2$. To walk from (3,1) to (1,2) we walk up one and left two. We repeat this walk starting at (1,2) to find the reflection point of (3,1), which is (1-2,2+1)=(-1,3).

25) $2\sqrt{7}$ Converting to Cartesian coordinates, (2, 30°) and (4, 150°) become $(2\cos 30Y, 2\sin 30Y) = (\sqrt{3}, 1)$ and $(4\cos 150Y, 5\sin 150Y) = (-2\sqrt{3}, 2)$ respectively. Thus, the distance between the two points is $\sqrt{(3\sqrt{3})^2 + (-1)^2} = \sqrt{28} = 2\sqrt{7}$. Alternatively, form the triangle using the origin and the two given points. The angle at the origin measures 120°, so using the law of cosines, the square of the distance between the points is $2^2 + 4^2 - 2 ? 2 ? 4 ? \cos 120Y = 4 + 16 - 16 ? (-\frac{1}{2}) = 28$.

26) $f^{-1}(x) = \frac{x-5}{2}$ Solving $y = 2x + 5$ for *x* yields the solution when we replace *x* by $f^{-1}(x)$ and *y* by *x*.

27) $m\Delta TMP = 130Y$ ΔTMP and ΔMPN are supplementary angles since they are interior angles on the same side of the transversal $\overline{MP}$ of the parallel lines $\overline{MT}$ and $\overline{NP}$. Thus, $m\Delta TMP = 180Y - m\Delta MPN = 180Y - 50Y = 130Y$.

28) 5 As $x \to \infty$, $f(x)$ behaves like $\frac{x^2}{2x^2} = \frac{1}{2}$, so $a = \frac{1}{2}$. The denominator of $f(x)$ is $2x^2 - 5x + 3 = (2x-3)(x-1)$ which equals 0 if x is $\frac{3}{2}$ or 1. Neither of these values of x make the numerator 0, so $\{b,c\} = \{\frac{3}{2},1\}$ and $\frac{b+c}{a} = \frac{3/2+1}{1/2} = \frac{5/2}{1/2} = 5$.

29) 14 Let S be the number of sessions Sleek attended and B be the number of sessions Barnes attended. Then $S = 2B$.
$55 = 13 + 10 + 11 + S + B = 34 + 2B + B \Rightarrow 21 = 3B \Rightarrow B = 7 \Rightarrow S = 14$.

30) a Note c is the identity element for this operation. Thus, you need to find the element x such that $a * x = c = x * a$. From the table we see $x = a$.

31) 6 sides Each segment connecting pairs of vertices of a polygon is either a side of the polygon or a diagonal. The number of diagonals of a polygon is the number of pairs of vertices minus the number of sides of the polygon, or
$\binom{n}{2} - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$. Christman's triangle has 0 diagonals, Cohen's quadrilateral has 2 diagonals, Bunn's pentagon has5 diagonals, Meuser's octagon has 20 diagonals, and Huffman's n-sided polygon has $\frac{n(n-3)}{2}$ diagonals. The statement regarding the diagonals of Meuser's polygon translates to
$20 = \frac{1}{2}(0 + 2 + 5 + \frac{n(n-3)}{2}) + 12 \Rightarrow 16 = 7 + \frac{n(n-3)}{2} \Rightarrow 18 = n(n-3) \Rightarrow n = 6$.

32) $a = 200, b = 3$ $\begin{pmatrix} 2 & 203 \\ 6 & 11 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & b \end{pmatrix} = \begin{pmatrix} a\cdot 0 + 1\cdot 2 & a\cdot 1 + 1\cdot b \\ 2\cdot 0 + 3\cdot 2 & 2\cdot 1 + 3\cdot b \end{pmatrix} = \begin{pmatrix} 2 & a+b \\ 6 & 2+3b \end{pmatrix}$. So,
$203 = a + b$ and $11 = 2 + 3b \Rightarrow b = 3$. Thus, $203 = a + 3 \Rightarrow a = 200$.

33) 60 minutes Each inlet pipe fills the pool at a rate of $\frac{1}{20}$ pools/minute. The outlet pipe drains the pool at a rate of $\frac{1}{12}$ pools/minute. We need the time t in minutes such that $\left(\frac{1}{20} + \frac{1}{20} - \frac{1}{12}\right)t = 1\,(\text{pool}) \Rightarrow \frac{1}{60}t = 1 \Rightarrow t = 60$ minutes.

34) 48π square units Let R be the radius of the outer circle. Then $2\pi R = 16\pi \Rightarrow R = 8$. The area of the ring is the area of the large circle minus the area of the small circle, or $\pi \cdot 8^2 - 16\pi = 48\pi$ square units.

35) 16 Note $\frac{a+b}{b} = \frac{a}{b} + 1$ which will have its greatest possible value when a is as large as possible and b is as small as possible. Thus, the greatest possible value is $\frac{75}{5} + 1 = 16$.

36) $\sqrt{5}$ The midpoint of $\overline{AB}$ is $\frac{1}{2}(-7+1, 4+6) = (-3,5)$. The equation of the circle can be rewritten as $(x+5)^2 + (y-4)^2 = 25 + 25 + 16$, and thus has center $(-5,4)$. The length of the segment connecting these points is
$\sqrt{(-3-(-5))^2 + (5-4)^2} = \sqrt{4+1} = \sqrt{5}$.

37) $x = 500$ Note $\log_x 2x = \dfrac{\log_{10} 2x}{\log_{10} x}$. Thus,

$$(\log_x 2x)(\log_{10} x) = \left(\frac{\log_{10} 2x}{\log_{10} x}\right)(\log_{10} x) = \log_{10} 2x$$ and we must solve $\log_{10} 2x = 3$ for x. Then $10^3 = 2x \Rightarrow 1000 = 2x \Rightarrow x = 500$.

38) Thales of Miletus Thales has both stories told about his great practical skills and also about him being an unworldly dreamer. Aristotle, for example, relates a story of how Thales used his skills to deduce that the next season's olive crop would be a very large one. He therefore bought all the olive presses and then was able to make a fortune when the bumper olive crop did indeed arrive. On the other hand Plato tells a story of how one night Thales was gazing at the sky as he walked and

fell into a ditch. A pretty servant girl lifted him out and said to him "How do you expect to understand what is going on up in the sky if you do not even see what is at your feet". As Brumbaugh says, perhaps this is the first absent-minded professor joke in the West! (Source: http://www-groups.dcs.st-and.ac.uk:80/~history/)

39) 25 cm — Let P be the pressure, T be the temperature, and V be the volume. Then there exists a constant k such that $P = k ? \frac{T}{V}$. Using the information in the 2nd sentence, $500 = k ? \frac{300}{30}$? $k = 50$. Thus, $P = 50 ? \frac{T}{V}$ and noting the units of measurement in the question did not change, $P = 50 ? \frac{285}{570} = 25$ centimeters.

40) 60 mph — Let r be the rate in mph that Charlie drove and let t be the number of hours she drove. Then $180 = rt$. If r is decreased by 6 mph, then t is increased by $\frac{1}{3}$ hour. Thus, $180 = (r-6)(t+\frac{1}{3})$? $180 = (r-6)(\frac{180}{r}+\frac{1}{3})$? $540r = (r-6)(540+r)$? $540r = 540r + r^2 - 3240 - 6r$? $r^2 - 6r - 3240 = 0$? $(r-60)(r+54)$? $r = 60$, since r must be greater than 0.

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