2017 OHMIO Pressure Round

1. An acute angle is formed by two lines of slope 1 and 7. Another line bisects this angle. What is its slope?

2. Find the sum of the **ten** smallest positive integers n for which $\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{n+}}}}$ ... is itself an integer.

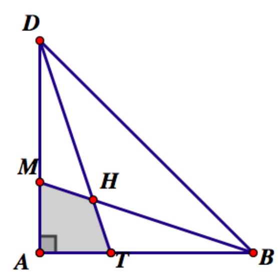
3. Find the volume enclosed within $|x|+|y|+|z|=3$.

4. A weird calculator has only two buttons as shown below. If the calculator currently displays 11, what is the least number of button presses needed to display 25?

Button A: Multiply displayed value by 2
Button B: Subtract 3 from the displayed value

5. In trapezoid *ABCD* with *AB* parallel to *CD*, the diagonals intersect at E. The area of triangle *ABE* is 32 and the area of triangle *CDE* is 50. Find the area of *ABCD*.

6. ΔDAB is a right isosceles triangle with points *T* and *M* trisecting their respective legs. Let *H* be the intersection of the segments connecting each trisection point to the opposite vertex of the triangle. What *fraction* of the area of ΔDAB is the area of quadrilateral *MATH*?



7. Three fair six-sided dice are rolled whose faces are one of 1, 2, 3, 4, 5, or 6. As a simplified fraction, find the probability the **sum** of the three values rolled is 10.

8. Find the positive integer n for which $\sum_{k=1}^{n}\lfloor\log_2 k\rfloor=153$ where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

9. Determine the value of $x^{17}+x^{-17}$ if x satisfies the equation $x^2+x+1=0$.

10. For $0\le x\le\pi$, $0\le y\le\pi$, find the number of real ordered pairs (x, y) that satisfy the system:

$$\begin{cases}\sin(x+y)=0\\ \sin(x-y)=0\end{cases}$$

Answers

1. 2	**2.** 440	**3.** 36	**4.** 7	**5.** 162
6. 1/6	**7.** 1/8	**8.** 42	**9.** -1	**10.** 5

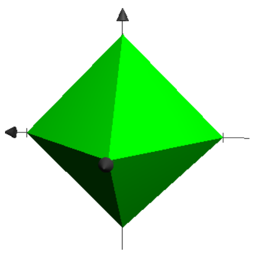
Solutions

1. *Solution 1:* Note $\tan q = m$ for a line of slope m intersecting the x-axis at an angle q. Let C be the slope of the angle bisector. We have $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$, so $\frac{7 - C}{1 + 7C} = \frac{C - 1}{1 + C}$ or $0 = (2C^2 - 3C - 2)$. We see $(2C + 1)(C - 2) = 0$. So C = **2.**

Solution 2: The line through O(0,0) and A(1,7) has slope 7, and OA = $5\sqrt{2}$. The line through O(0,0) and B(5,5) has slope 1 and OB = $5\sqrt{2}$. Thus the slope of the bisector of angle AOB is the slope of a perpendicular to line AB, which is $\frac{7-5}{1-5} = -\frac{1}{2}$ so the slope of the bisector is **2**.

2. If $k = \sqrt{n + \sqrt{n + \sqrt{n + \sqrt{n +}}}} \ldots$, then $k^2 = n + k$ so $k = \frac{1+\sqrt{1+4n}}{2}$ which is an integer when n = 2, 6, 12, … or n = 1*2, 2*3, 3*4,…10*11. The sum of these values is $\sum_{k=1}^{10} k(k+1) = \sum_{k=1}^{10} k^2 + \sum_{k=1}^{10} k = \frac{10*11*21}{6} + \frac{10*11}{2} = 385 + 55 = \mathbf{440.}$

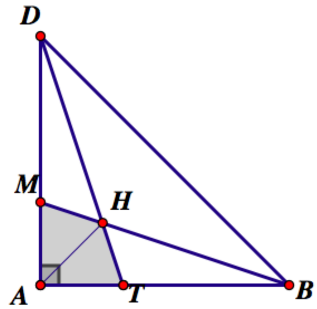
3. This graph produces an octahedron, which is two square pyramids joined by a common base. The area of the base is $(3\sqrt{2})(3\sqrt{2}) = 18$ and the volume of the octahedron is 2(18)(3)/3 = **36**



4. We can backtrack starting with 25, excluding a number in parenthesis if the number was obtained in less steps. We see that it takes **7** steps to access 11.
0: 25
1: 28
2: 14 or 31
3: 7, 17, or 34
4: 10, 20, (17), 37
5: 5, 13, (10), 23, 40
6: 8, 16, 26, (20), 43
7: 4, 11, (8), 19, (13), 29 or 46

5. By Angle Angle, *ABE* ~ *CDE*. Let *F* and *G* be the foot of the perpendicular when *E* is extended to *AB* and *CD*, respectively. Then *EG*/*FE* = *CD*/*AB* = *k*. Then (*AB*)(*FE*) = 64 and (*DC*)(*EG*) = 100. So 100/64 = k^2, so *k* = 1.25. So area of *ABCD* = ½(*AB* + *DC*)(*FG*) or ½(*AB* + 1.25*AB*)(*FE* + 1.25*FE*) = ½(2.25)(2.25)*(AB)(FE)* = ½(2.25)(2.25)(64) = **162.**

6. Without loss of generality, suppose $AB = AD = 3$. The area of ΔDAB is ½(3)(3) = 9/2. The area of triangle MAB is ½(3)(1) = 3/2. Let y = area of ΔHTB. Since $TB = 2AT$, area $\Delta AHT = y/2$ as they have the same height. Thus, the area of $MATH = y$ and $2y = 3/2$, so $y = ¾$. Thus, the ratio of areas is (3/4)/(9/2) = **1/6**.



7. *Solution1 :* Each roll has probability $1/6^3 = 1/216$. Disregarding order, the possible triples for a sum of 10 are: (1, 3, 6), (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4), (3, 3, 4).
Those triples without any repeats can be permuted in 3! = 6 ways and those with a duplicate value can be permuted in 3!/2 = 3 ways. This yields 3(6) + 3(3) = 27 possible sums of 10 with a probability of 27/216 = **1/8**.

Solution 2: The 1st die can roll 1, 2, 3, 4, 5, or 6. The 2nd and 3rd dice must correspondingly roll a total of 9, 8, 7, 6, 5, or 4 to complete the 10 total (on 3 dice). There are 4, 5, 6, 5, 4, and 3 ways to roll those totals on the 2nd/3rd dice, for a total of 4+5+6+5+3+3 = 27 ways to accomplish the 10 total. The probability as before is 27/216 = **1/8**.

8. Note $\lfloor \log_2 1 \rfloor = 0, \lfloor \log_2 2 \rfloor = \lfloor \log_2 3 \rfloor = 1, \lfloor \log_2 4 \rfloor = \lfloor \log_2 5 \rfloor = \lfloor \log_2 6 \rfloor = \lfloor \log_2 7 \rfloor = 2$, etc.
Let $S_n = \sum_{k=1}^{2^n-1} \lfloor \log_2 k \rfloor = 1(0) + 2(1) + 4(2) + 8(3) + \cdots + 2^{n-1}(n-1)$.
Then $S_1 = 0, S_2 = 2, S_7 = 10, S_{15} = 34, S_{31} = 98, S_{63} = 258$.
Thus, $5m + 98 = 153$ so $m = 11$. Thus $n = 31 + 11 =$ **42**.

9. *Solution1:* Since x is nonzero, $x^2 + x + 1 = 0$ implies $x^1 + x^{-1} = -1$. Multiplying the first equation by x, $x^3 + x^2 + x = 0$ or $x^3 = -(x^2 + x) = -(-1) = 1$.
Thus, $x^{17} + x^{-17} = (x^3)^5 x^2 + (x^3)^{-5} x^{-2} = x^2 + x^{-2} = (x^1 + x^{-1})^2 - 2 = (-1)^2 - 2 =$ **-1.**

Solution 2: Since $x^2 + x + 1 = 0$, we know x is a root of $(x-1)(x^2 + x + 1) = 0$ or $x^3 = 1$. By De Moivre's Theorem, $x = cis\frac{\pm 2\pi}{3}$ where "cis" stands for "cosine + i*sine". Then $x^{17} = cis\frac{\pm 34\pi}{3} = cis\frac{\pm 4\pi}{3} = cis\frac{\mp 2\pi}{3}$. Now note $x^{-17} = cis\frac{\mp 34\pi}{3} = cis\frac{\mp 4\pi}{3} = cis\frac{\pm 2\pi}{3}$. So x^{17} and x^{-17} are complex conjugates and their sum is $2cos\frac{2\pi}{3} = -1$.

10. The first equation allows us to say $x + y = \pi n$ for some integer n and the second yields $x - y = \pi m$ for some integer m. Using elimination, $x = \frac{\pi(m+n)}{2}$ and $y = \frac{\pi(n-m)}{2}$. Given the constraints on x and y, $0 \le m + n \le 2$ and $0 \le n - m \le 2$ for integers m, n. We note that n can only take on values of 0, 1, or 2. If $n = 0$ then $m = 0$ only. If $n = 1$, then $m = -1$, 0, or 1. If $n = 2$, then $m = 0$. This yields **5** solutions for (x, y) : (0, 0), $(0, \pi)$, $(\frac{\pi}{2}, \frac{\pi}{2})$, $(\pi, 0)$, (π, π). This is confirmed in a graph.