2017 OHMIO Individual Competition

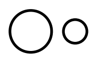
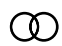
1. On a winter hike with friends (all of whom were wearing either a scarlet or gray hat), I saw twice as many scarlet hats as gray. “That’s silly,” said a friend. “I see the same number of scarlet hats as gray!” How many of us are hiking?

2. For bases a and b, $144_a = 441_b$. Find the minimum value of $a \cdot b$.

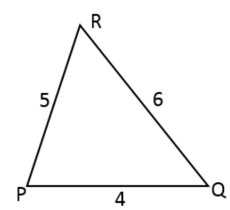
3. A particle travels in the coordinate plane such that $x = 3sin(t)$ and $y = -2cos(t)$, for $0 \leq t \leq 2\pi$. The particle starts and ends at the point $(0, -2)$, and sweeps out a closed shape. What is the exact area of the interior of that shape?

4. In $\Delta ABC, m\angle A = 60^{\circ}, m\angle B = 45^{\circ}$, and $AC = 4$. Length AB can be expressed as $p + q\sqrt{r}$ for integers p, q, and r. Find pqr.

5. Which of the following figures could NOT be the intersection of a plane and an ordinary torus?

a)  b)  c)  d) 

6. In ΔPQR, shown at right, B is on $\overline{QR}$ such that $\overline{PB}$ bisects ∠P. Find PB.



7. A semicircle is cut from a rectangular sheet of paper 6 cm by 12 cm (with its diameter containing the long edge), then rolled into a cone. What is the exact volume of the cone?

8. The polynomial $f(x) = x^3 + kx^2 + 66x + m$ has three distinct positive integer roots that are in an arithmetic progression. Calculate the value of m.

9. Every morning, a boy would bring the newspaper to his grandmother, who would thank him by giving him a coin. She was equally likely to give him a nickel or a quarter. The boy saved his money, then when he saved at least 25 cents, he would spend it all on candy. On such a candy day, what is the probability that he has exactly 25 cents to spend? Express as a fraction of relatively prime integers.

10. Find all real x satisfying $\begin{vmatrix} x-1 & 4-4x & x-1 \\ 1-x & 2x-2 & 1-x \\ 5x-5 & \frac{1}{2}-\frac{1}{2}x & 3x-3 \end{vmatrix} = 32.$

11. What is the shortest distance from the point $(6,5)$ to the line $4x + 3y = -11$?

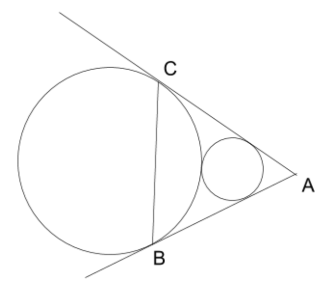
12. An equilateral triangle is inscribed in a unit square (i.e. one whose side is length 1) such that one vertex is on a corner of the square, and the opposite side is perpendicular to a diagonal of the square. The area of the triangle can be expressed as $a + b\sqrt{c}$ for integers a, b, and c. Find abc.

13. Find all positive solutions for a: $(4a + 6)^2 = a^3 + (a + 1)^3 + (a + 2)^3 + (a + 3)^3$

14. Take a rectangular piece of paper 12 cm by 16 cm and fold one corner to the diagonally opposite corner. What is the area of the pentagon thus created?

15. Simplify: $log_{0.4}(log_3 9^3 - log_{16} log_4 2)$

16. Two externally tangent circles have common external tangents which touch the larger circle at C and B as shown. If the circles have radii of 12 and 3, what is CB?



17. $ln15 \approx 2.708$, $ln18 \approx 2.890$, and $ln30 \approx 3.401$. To the nearest thousandth, find $ln25$.

18. a, b, c, and d form an arithmetic sequence of two-digit positive integers. a and c are squares; b and d are primes. The average of all four numbers also happens to be a prime. What is that average?

19. The number 1230045067000 has three "trailing zeros." How many trailing zeroes does $\binom{201}{42}$ have?

20. A line through the origin is tangent to $f(x) = -4x^2 + 16x - 9$ at a point in the first quadrant. Find the point of tangency.

TB1. Compute $\frac{1+2+3+\cdots+n}{\sqrt{1+3+5+\cdots+(2n-1)}}$ for $n = 2017$.

TB2. 2017 is prime, and its digits sum to 10. Determine the largest prime less than 2017 that also has digits that sum to 10.

2017 OHMIO Individual Competition ANSWERS

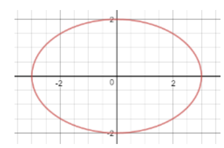
1. 7
2. 45
3. 6π
4. 12
5. c
6. $\frac{10}{3}$
7. $9\pi\sqrt{3}$ cm^3
8. -80
9. $\frac{17}{32}$
10. 3
11. 10
12. -18
13. 1
14. 117
15. -2
16. $\frac{96}{5}$ or 19.2
17. 3.219
18. 43
19. 2
20. $\left(\frac{3}{2}, 6\right)$

TB1: 1009

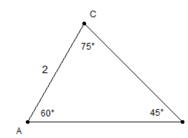
TB2: 1801

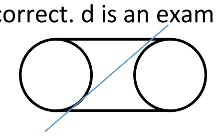
2017 OHMIO Individual Competition SOLUTIONS

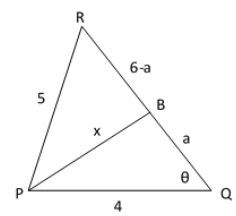
1. Let S = # of scarlet hats in the group, and G = # of gray hats in the group. Since I see more scarlet hats than my friend, I must be wearing gray. I see G – 1 gray and S scarlet hats, so 2(G – 1) = S. My friend sees S – 1 scarlet and G gray hats, so S – 1 = G. Combining, 2(S – 1 – 1) = S, then S = 4 and G = 3, and there are **7** friends hiking.
2. First, note that $a > b > 4$.
$a^2 + 4a + 4 = 4b^2 + 4b + 1 \rightarrow (a+2)^2 = (2b+1)^2 \rightarrow a + 2 = 2b + 1$.
Try $b = 5$. Then $a = 9$, and $a \cdot b = \mathbf{45}$. $b > 5$ makes $a > 9$, so $a \cdot b = \mathbf{45}$ is the minimum.
3. The parametric equations produce an ellipse with vertices $(\pm 3, 0)$ and covertices $(0, \pm 2)$.
The area of the ellipse is $ab = \pi(3)(2) = \boldsymbol{6\pi}$.



4. By Law of Sines, $\frac{AB}{sin75^\circ} = \frac{4}{sin45^\circ}$, so , $AB = \frac{4sin75^\circ}{sin45^\circ} = \frac{4\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)}{\frac{\sqrt{2}}{2}} = \frac{2(\sqrt{2}+\sqrt{6})}{\sqrt{2}} = 2 + 2\sqrt{3}$.
So $p = 2$, $q = 2$, and $r = 3$, and $pqr = \mathbf{12}$.
$sin75^\circ = sin(30^\circ + 45^\circ) = sin30^\circ cos45^\circ + cos30^\circ sin45^\circ = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$.



5. a, b, and d are correct. d is an example of "Villarceau circles," where the plane is positioned thus:  Here's a view of the cross-section: 
6. By Triangle Angle-Bisector Theorem, $\frac{5}{6-a} = \frac{4}{a}$, so $a = \frac{8}{3}$.
By Law of Cosines, $os\theta = \frac{4^2+6^2-5^2}{2\cdot4\cdot6} = \frac{9}{16}$.
Also, $x^2 = 4^2 + a^2 - 2\cdot 4\cdot a\cdot cos\theta = 16 + \left(\frac{8}{3}\right)^2 - 8\left(\frac{8}{3}\right)\left(\frac{9}{16}\right) = \frac{100}{9}$. Thus $B = x = \frac{\mathbf{10}}{\mathbf{3}}$.



7. The radius of the semicircle is 6, so the arc length of the semicircle is $\frac{1}{2}(2\pi r) = 6\pi$.
Thus the circumference of the base of the cone is 6π, and the radius of the base is $\frac{6\pi}{2\pi} = 3$.
The slant height of the cone equals the radius of the semicircle = 6.
The height of the cone satisfies $h^2 + 3^2 = 6^2$, so $h = 3\sqrt{3}$. Finally, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 3^2(3\sqrt{3}) = \boldsymbol{9\pi\sqrt{3}}$.
8. For roots p, q, and r, $(x-p)(x-q)(x-r) = x^3 - (p+q+r)x^2 + (pq+pr+qr)x - pqr$.
Since the roots are in arithmetic progression, write them as $r - a$, r, and $r + a$.
So the coefficient of x is $(r-a)r + (r-a)(r+a) + r(r+a) = 3r^2 - a^2 = 66$.
The roots are positive integers, so $r \geq 5$ and $0 < a < r$. Try $r = 5$. $3(5)^2 - a^2 = 66 \rightarrow a = 3$, so the roots could be 2, 5, and 8. $r > 5 \rightarrow a > r$, so $r = 5$ is unique. Thus $m = -(2)(5)(8) = \mathbf{-80}$.
9. There are exactly two events that are successful: five nickels (on days 1-5), and a quarter on day 1.
Getting a quarter after 1, 2, 3, or 4 nickels earns the candy day, but not with exactly 25 cents.
Since the two successful events are mutually exclusive, add their probabilities.
$P(five\ nickels) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$. $P(one\ quarter) = \frac{1}{2}$. $P(either) = \frac{1}{32} + \frac{1}{2} = \frac{\mathbf{17}}{\mathbf{32}}$.
10. $\begin{vmatrix} x-1 & 4-4x & x-1 \\ 1-x & 2x-2 & 1-x \\ 5x-5 & \frac{1}{2}-\frac{1}{2}x & 3x-3 \end{vmatrix} = (x-1)^3 \begin{vmatrix} 1 & -4 & 1 \\ -1 & 2 & -1 \\ 5 & -\frac{1}{2} & 3 \end{vmatrix}$
$$= (x-1)^3\left[\left(1\cdot2\cdot3 + (-4)(-1)5 + 1(-1)\left(-\frac{1}{2}\right)\right) - \left(5\cdot2\cdot1 + \left(-\frac{1}{2}\right)(-1)1 + 3(-1)(-4)\right)\right]$$
$$= (x-1)^3\left[\left(6 + 20 + \frac{1}{2}\right) - \left(10 + \frac{1}{2} + 12\right)\right] = (x-1)^3(4) = 32 \rightarrow (x-1)^3 = 8 \rightarrow x = \mathbf{3}$$
11. $4x + 3y = -11$ has slope $-\frac{4}{3}$, so the slope of its perpendicular is $\frac{3}{4}$.
The equation of the perpendicular through (6,5) is $y - 5 = \frac{3}{4}(x - 6)$, or $3x - 4y = -2$.

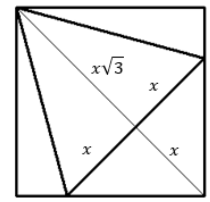
Solving $4x + 3y = -11$ and $3x - 4y = -2$ finds the foot of the perpendicular at $(-2, -1)$.

The distance from $(6,5)$ to $(-2,-1)$ is $\sqrt{\left(6-(-2)\right)^2 + \left(5-(-1)\right)^2} = \sqrt{(8)^2 + (6)^2} = \mathbf{10}$

Alternative solution: The distance from point (p, q) to line $ax + by + c = 0$ is $\frac{|ap+bq+c|}{\sqrt{a^2+b^2}}$,

so $d = \frac{|4\cdot 6+3\cdot 5+11|}{\sqrt{3^2+4^2}} = \frac{|24+15+11|}{\sqrt{25}} = \frac{50}{5} = \mathbf{10}$.

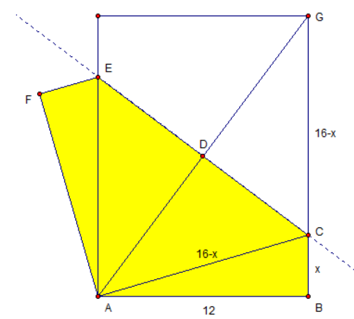
12. Draw the diagonal, and let x be half of the base of the triangle. $x\sqrt{3}$ is the height of the triangle. Since the diagonal and the base of the triangle are perpendicular, an isosceles right triangle is formed, and another segment of length x is found. Thus the diagonal can be expressed as $x + x\sqrt{3} = \sqrt{2}$ (since the square has unit length).



Solving, $x = \frac{\sqrt{2}}{1+\sqrt{3}} = \frac{\sqrt{6}-\sqrt{2}}{2}$. The area of the triangle is $x^2\sqrt{3} = \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2\sqrt{3} = 2\sqrt{3} - 3$.
Thus $a = -3$, $b = 2$, and $c = 3$, and $abc = \mathbf{-18}$.

13. $16a^2 + 48a + 36 = a^3 + a^3 + 3a^2 + 3a + 1 + a^3 + 6a^2 + 12a + 8 + a^3 + 9a^2 + 27a + 27$
$\rightarrow 16a^2 + 48a + 36 = 4a^3 + 18a^2 + 42a + 36 \rightarrow 4a^3 + 2a^2 - 6a = 0$
$\rightarrow 2a(2a^2 + a - 3) = 0 \rightarrow 2a(2a+3)(a-1) = 0$. So $a \in \left\{0, -\frac{3}{2}, 1\right\}$. The only positive solution is $a = \mathbf{1}$.
Note that $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$. Since $a + (a+1) + (a+2) + (a+3) = 4a + 6$, $(4a+6)^2 = a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3$ is a demonstration of the formula for $n = 4$ and $a = \mathbf{1}$.

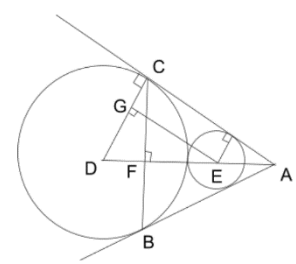
14. Let BC $= x$. Then $GC = AC = 16 - x$. $x^2 + 12^2 = (16-x)^2$, yielding $= \frac{7}{2}$.
$a\Delta ABC = \frac{1}{2}(12)\left(\frac{7}{2}\right) = 21 = a\Delta AFE$. Any line through D bisects the rectangle, so $aABCE = \frac{1}{2}(12)(16) = 96$.
$aABCEF = aABCE + a\Delta AFE = 96 + 21 = \mathbf{117}$.



Alternatively, find $AD = 10$ (since $AG = 20$, from 12-16-20 triangle). Since $x = \frac{7}{2}$, $AC = 16 - \frac{7}{2} = \frac{25}{2}$, so $CD^2 = \left(\frac{25}{2}\right)^2 - 10^2 = \frac{225}{4}$,
and $CD = \frac{15}{2}$. $a\Delta ACD = \frac{1}{2}\left(\frac{15}{2}\right)(10) = \frac{75}{2} = a\Delta ADE$.
$aABCEF = a\Delta ABC + a\Delta ACD + a\Delta ADE + a\Delta AFE = 21 + \frac{75}{2} + \frac{75}{2} + 21 = \mathbf{117}$.

15. $\log_{\frac{2}{5}}\left(\log_3(3^2)^3 - \log_{16}\frac{1}{2}\right) = \log_{\frac{2}{5}}\left(6 - \left(-\frac{1}{4}\right)\right) = \log_{\frac{2}{5}}\left(\frac{25}{4}\right) = \log_{\frac{2}{5}}\left(\frac{5}{2}\right)^2 = -\mathbf{2}$.

16. Draw radii from D and E to points of tangency and $\overline{GE}$ parallel to the tangent line. $GC = 3$, since it is equal to the radius of the smaller circle.
So $DG = 9$ and $DE = 15$, making $GE = 12$.
By AA similarity, $\Delta CDF \sim \Delta EDG$. So $\frac{CF}{EG} = \frac{CD}{ED} \rightarrow \frac{CF}{12} = \frac{12}{15} \rightarrow CF = \frac{12\cdot 12}{15} = \frac{48}{5}$,
and $CB = 2CF = \frac{\mathbf{96}}{\mathbf{5}} = \mathbf{19.2}$.



17. $25 = 5 \cdot 5 = \frac{3\cdot 5\cdot 5\cdot 6}{3\cdot 6} = \frac{15\cdot 30}{18}$.
$\ln 25 = \ln\left(\frac{15\cdot 30}{18}\right) = \ln 15 + \ln 30 - \ln 18 = 2.708 + 3.401 - 2.89 = \mathbf{3.219}$.

18. We need the average of two squares to be a prime. If the squares are even, their average will also be even (since each square is a multiple of 4). Thus we only need to search the two-digit odd squares. Consider 25, 49, and 81. The average of 49 and 81 is 65, a composite. The average of 25 and 81 is 53, a prime. The average of 25 and 49 is 37, also prime. In the first prime case, the common difference is 28, yielding d=109, which has three digits. In the second case, the common difference is 12, yielding $a = 25$, $b = 37$, $c = 49$, and $d = 61$, whose average is **43**.

19. $\binom{201}{42} = \frac{201!}{42!159!} = \frac{201\cdot 200\cdot 199\cdots 160}{1\cdot 2\cdot 3\cdots 42}$. Trailing zeros come from factors of five (since twos are plentiful). Fives in the numerator come from 160, 165, 170, 175, 180, 185, 190, 195, and 200, where 175 and 200 contribute 2 each, for a total of 11 fives. From the denominator we get fives from 5, 10, 15, 20, 25, 30, 35, and 40 (with 2 from 25), for a total of 9 fives. Canceling, $11 - 9 = 2$ fives, so there are **2** trailing zeros.

20. $f(x) = -4x^2 + 16x - 9$, and $f'(x) = -8x + 16$. Let the point of tangency be (a, b). The line through (a, b) and the origin has a slope of $\frac{b}{a}$. $f(a) = -4a^2 + 16a - 9 = b$. $f'(a) = -8a + 16 = \frac{b}{a} = \frac{-4a^2+16a-9}{a}$. $-8a^2 + 16a = -4a^2 + 16a - 9 \rightarrow 4a^2 = 9 \rightarrow a = \frac{3}{2} \rightarrow b = f\left(\frac{3}{2}\right) = 6$. The point of tangency is $\left(\mathbf{\frac{3}{2}}, \mathbf{6}\right)$. Note that there is a second point of tangency, found where $a = -\frac{3}{2} \rightarrow b = f\left(-\frac{3}{2}\right) = -42$, making the other point of tangency $\left(-\frac{3}{2}, -42\right)$.

TB1: $\frac{1+2+3+\cdots+n}{\sqrt{1+3+5+\cdots+(2n-1)}} = \frac{\frac{n(n+1)}{2}}{\sqrt{n^2}} = \frac{n+1}{2}$. For $n = 2017$, $\frac{n+1}{2} = \frac{2017+1}{2} = \mathbf{1009}$.

TB2: Working backward, we have 1900 (not prime) and **1801**. Testing prime factors up to 41 will show it is prime.