OHMIO 2017 Ciphering Solutions

Answers

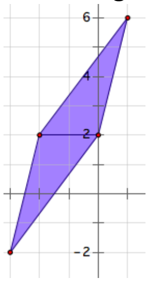
Practice 5/2	**1.** 8	**2.** 3/64	**3.** $\sqrt{2}$	**4.** -17/8	**5.** 4/105
6. 15	**7.** 2π	**8.** 24	**TB1.** π	**TB2.** 807	

Solutions

Practice. *Solution 1:* By substitution, $(4^{a_1})^{a_2} = 6$. By substituting again, $((4^{a_1})^{a_2})^{a_3} = 7$, and so on. Thus, $(((4^{a_1})^{a_2})^{a_3\cdots})^{a_{28}} = 32$, and thus $4^{a_1 a_2 a_3 \ldots a_{28}} = 32$ and hence $2a_1 a_2 a_3 \ldots a_{28} = 5$ by equating bases. So $a_1 a_2 a_3 \ldots a_{28} = \frac{5}{2}$

Solution 2: We know $\log_4 5 = a_1$, $\log_5 6 = a_2$, … $\log_{31} 32 = a_{28}$.
Then $a_1 a_2 a_3 \ldots a_{28} = \log_4 5 \log_5 6 \ldots \log_{31} 32 = \log_4 32 = \frac{5}{2}$ by Change of Base.

1. A quick sketch shows these points make a parallelogram. One way to find this area is to note that two congruent triangles with base 2 and height 4 are appended, each with area ½(2)(4) = 4. Thus, the area is 8. One could also find the area using a cross product or Pick's Theorem.



2. The slope of $x + 2y = 1$ is -1/2. For no solution, we need $Ax + By = 3$ to have slope -1/2, however, we do not want 3*x* + 6*y* = 3 as this would create the same line and thus infinitely-many solutions. Hence, we could have (*A, B*) = (1, 2), (2, 4), or (4, 8) allowing 3 possibilities. By independence, there are (8)(8) = 64 possible pairs of (*A, B*) yielding 3/64 as the probability.

3. The area can be found as $\frac{1}{2}(4\sqrt{6})(4\sqrt{3}) = 24\sqrt{2}$. When the diagonals are drawn, we get four 30-60-90 triangles with each hypotenuse measuring 6 so the perimeter is 24. Then $\frac{x}{y} = \frac{24\sqrt{2}}{24} = \sqrt{2}$.

4. Note that $csc^2x - cot^2x = 1$ and $1 + tan^2x = sec^2x$. Thus, $f(x) = \frac{17sin^3xcosx}{sec^2x} = 17(sinxcosx)^3$. By the double angle identity, $sinxcosx = \frac{1}{2}sin2x$ and thus,
$f(x) = \frac{17}{8}(sin2x)^3$. The minimum of $sin2x$ is -1 so the minimum of *f* is $\frac{-17}{8}$.

5. Any path from A to B will involve 6 rights and 4 ups for 10 total moves. There are a total of $10C4 = \frac{10!}{6!4!} = 210$ paths from A to B. There are 4C1 = 4 paths from A to X and
6C3 = 20 paths from X to B. Of these, 2C1 = 2 paths go from X to Y and 4C2 = 6 paths go from Y to B. Thus, there are (2)(6) = 12 paths from X to B through Y leaving 20 – 12 = 8 paths from X to B not through Y. The desired probability is then 8/210 = 4/105.

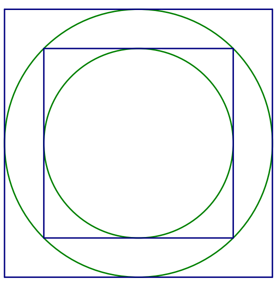
6.

$$(1)\ x - y + z = 1$$
$$(2)\ y - z + u = 2$$
$$(3)\ z - u + v = 3$$
$$(4)\ u - v + x = 4$$
$$(5)\ v - x + y = 5$$

Solution 1:
Add equations 1 and 2: $x + u = 3$. Now subtract equation 4 to get *v* = -1.
Add equations 2 and 3: $y + v = 5$. Now subtract equation 5 to get *x* = 0.
Substituting, *u* = 3, *y* = 6 and *z* = 7. The sum is *x* + *y* + *z* + *u* + *v* = 0 + 6 + 7 + 3 + -1 = 15.

Solution 2:
Add all 5 equations together to get *x* + *y* + *z* + *u* + *v* = 1 + 2 + 3 + 4 + 5 = 15.

7. Let's look at the first few iterations. The radius of the largest circle is 2/1 = 1 and the circle's area is π. The diameter of this circle is a diagonal of the next square, so the side length of the next square is $\frac{2}{\sqrt{2}} = \sqrt{2}$. The radius of the next circle is then $\frac{\sqrt{2}.}{2}$ and its area is $\frac{\pi}{2}$. Continuing, we see each circle's area is ½ that of the previous area. The infinite sum of circle areas is then $\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = 2\pi$.



8. We can write $f(x) = (x - log_4 8)(x - log_8 4) = \left(x - \frac{3}{2}\right)\left(x - \frac{2}{3}\right) = x^2 - \frac{13}{6}x + 1$. Then *f*(6) = 36 – 13 + 1 = 24.

Tie-Breaker 1. The volume required is $\pi \int_0^1 (\frac{e^x + e^{-x}}{2})^2 - (\frac{e^x - e^{-x}}{2})^2 dx = \pi \int_0^1 1\, dx = \pi$

Tie-Breaker 2. Among the first 12 positive integers, there are 4 multiples of 3 and 3 multiples of 4, double-counting 12 so there are 3 + 4 – 1 = 6 such multiples of 3 or 4. Thus, from 1 to 60 there are 30 multiples of 3 or 4; however, of these 15, 20, 30, 40, 45, and 60 are multiples of 5 leaving 30 – 6 = 24 desired numbers from 1 to 60. Since 1980/60 = 33, we have accounted for 24(33) = 792 such numbers. We can now just count directly from the remaining numbers. There are 15 such numbers for a total of 807.