2006 OHMIO PRESSURE ROUND SOLUTIONS

1) A speaker talked for sixty minutes to a full auditorium. Ten percent of the audience heard the entire talk, and twenty percent slept through the entire talk. Half the remainder heard one-third of the talk, and the other half heard two-thirds of the talk. What was the average number of minutes of the talk heard by members of the audience?

Solution: *It is not necessary to know the number of people in the audience … but for the purposes of this solution, suppose there are N people in the audience. Then by the given information:*

0.10N listened for 60 minutes, 0.20N listened for 0 minutes,
0.35N listened for 20 minutes, 0.35N listened for 40 minutes.

That means that the total number of minutes listened was

$$(0.10N)(60) + (0.20N)(0) + (0.35N)(20) + (0.35N)(40) = 27N$$

So the average number of minutes was (27N)/(N) = ***27 minutes****. Observe that the use of N was not necessary; the answer could have been found by simply computing*

$$(0.10)(60) + (0.20)(0) + (0.35)(20) + (0.35)(40) = \mathbf{27\ minutes}.$$

2) $\sin 30^0 + \tan 45^0 + \cos 60^0 = \sec A$, $0^0 \leq A < 360^0$. **Find all values of A.**

Solution: *Either from memory, or by drawing reference triangles, we know that* $\sin 30^o = \cos 60^o = 0.5$, *and* $\tan 45^o = 1$, *so we need to solve sec A = 2. The secant function is the reciprocal of the cosine, so we seek angle A so that cos A = 0.5. We already observed that* 60^o *is one such angle; the other (between 0 and 360) is* 300^o*. Therefore,*

A = 60, 300 or $60^o, 300^o$

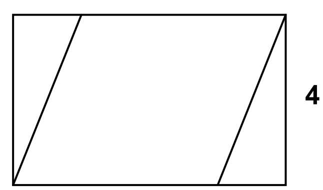
3) Factor completely over the reals: $a^4 + a^2b^2 + b^4$ **.**

Solution: *Add and subtract* a^2b^2*, which yields* $a^4 + 2a^2b^2 + b^4 - a^2b^2$*. The first three terms are the perfect square* $(a^2 + b^2)^2$*, and of course* a^2b^2 *is a perfect square, so this can be factored as a difference of squares:*

$$(a^2 + b^2 + ab)(a^2 + b^2 - ab).$$

4) What is the greatest possible area (in square inches) for a rhombus constructed on a 4 inch by 6 inch sheet of paper?

Solution: *To form a rhombus, we need to cut triangles off two diagonally opposite corners of the rectangle (see the picture on the right). We need choose x so that the hypotenuse of this triangle (with length* $\sqrt{4^2 + x^2}$ *) is equal in length to the "leftover" bottom of the rectangle (length 6 – x), so we need to solve*



$$16 + x^2 = (6 - x)^2.$$

This simplifies to 12x = 20, or x = 5/3, so the area of the rhombus is

$$A = (6-5/3)(4) = 24 - (4)(5/3) = 52/3 = 17\ 1/3$$

5) Softball player Berni Mac has 120 hits in 300 at-bats for a current batting average of .400. In today's game, she will have 5 at-bats. What is the probability that she will get *exactly* 2 hits?

Solution 1: *The probability of getting two hits followed by three outs is*

$$(0.4)(0.4)(0.6)(0.6)(0.6) = 2\times2\times3\times3\times3/5^5 = 108/3125.$$

But there are "5 choose 2" = 10 ways to have 2 hits in 5 at-bats, so the total probability is

1080/3125 or 216/625 or .3456

Solution 2: *Similarly, using the binomial expansion:*

$(a+b)^n$, *where a = success (getting a hit); b = failure; and n = times at bat.*

$(.4 + .6)^5 = .4^5$ (FIVE HITS) + $5(.4)^4(.6)^1$ (EXACTLY FOUR HITS) + $10(.4)^3(.6)^2$ (EXACTLY THREE HITS) + $\underline{10(.4)^2(.6)^3}$ (EXACTLY TWO HITS)….

$$\underline{10(.4)^2(.6)^3} = .3456$$

6) $\sum_{n=0}^{4}(n!) - \sum_{n=0}^{4} n = x^2 - 25$. Solve for all values of x.

Solution: $\sum_{n=0}^{4}(n!) = 0!+1!+2!+3!+4! = 1+1+2+6+24 = 34$, *and* $\sum_{n=0}^{4} n = 0+1+2+3+4 = 10$, *so we need to solve $x^2 - 25 = 24$, or $x^2 = 49$. That means that **$x = 7$ or $x = -7$.***

7) The sum of the squares of the roots of the equation $x^2 + 2kx = 6$ is 20. Find the lesser of the two possible values of k.

Solution 1: *Use the quadratic formula to solve $x^2 + 2kx - 6 = 0$, yielding the roots $r_1 = -k - \sqrt{k^2 + 6}$ and $r_2 = -k + \sqrt{k^2 + 6}$. As $r_1^2 + r_2^2 = 20$, we have $2[k^2 + (k^2 + 6)] = 20$, so $4k^2 = 8$, and $k = \pm\sqrt{2}$. The lesser of these two possible values is **$-\sqrt{2}$**.*

Solution 2: *Writing the equation in factored form, we have*

$$0 = (x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 + 2kx - 6.$$

Therefore,

$$r_1 + r_2 = -2k \quad \text{and} \quad r_1 r_2 = -6.$$

Squaring the first equation gives $r_1^2 + 2r_1 r_2 + r_2^2 = 4k^2$. Substitute $r_1^2 + r_2^2 = 20$ and $2r_1 r_2 = -12$ on the left side, which leaves $8 = 4k^2$, as in solution 1.

8) Solve for all real values of x: $5(4^x) = 4^{2x} + 4$.

Solution: *Let $z = 4^x$, so that this equation becomes $5z = z^2 + 4$. This is quadratic, and easily factored:*

$$0 = z^2 - 5z + 4 = (z - 1)(z - 4),$$

*so $z = 1$ or $z = 4$. Solving $4^x = 1$ or $4^x = 4$ for x gives **$x = 0$ or $x = 1$**.*

9) A lattice point (x,y) in the rectangular coordinate plane is a point in which both x and y are integers. How many lattice points are in both the <u>interior</u> of the circle $x^2 + y^2 - 4x + 2y = 11$, and <u>above</u> the line $y = x - 4$?

Solution: *Complete the squares in the first equation to find the center and radius of the circle:*

$$x^2 - 4x + 4 + y^2 + 2y + 1 = 11 + 4 + 1$$
$$(x - 2)^2 + (y + 1)^2 = 4^2$$

*So the circle is centered at $(2, -1)$ and has radius 4. Sketch this circle, and the line $y = x - 4$, and observe that there are **25 lattice points** inside the circle and above the line.*

10) The product of the ages of three children is 1428. The children's ages are in arithmetic progression. What are the three ages?

Solution: *1428 factors into $2\times 2\times 3\times 7\times 17$, so the ages of the three children must be made up of factors chosen from this list, meaning that the possible ages are:*

2, 4, 6, 7, 12, 14, 17, 21, …

As we were told these were "children," we presume that any higher ages are not allowed (and even 21 is probably too old). From this list of ages, the prime factor 17 only shows up as age 17, and only 7 and 12 would give an arithmetic progression, so the ages are:

7, 12, 17

(Perhaps the 17-year-old would prefer not to be called a "child," but that is not a mathematical issue.)