2006 OHMIO Individual Solutions

1. George's car gets 20 miles per gallon of gasoline. How many miles can he drive with the \$20.06 worth of gasoline that he bought at a price of exactly \$2.36 per gallon?

   Answer: 170, or 170 miles

   Solution: $\dfrac{20.06 \text{ dollars}}{2.36 \text{ dollars/gallon}}$ = 8.5 gallons purchased, so he can drive 8.5 gallons $\times \dfrac{20 \text{ miles}}{\text{gallon}}$ = 170 miles.

2. Find the length of the set of points on $x^2 + y^2 = 1$ that are closer than 1 unit from (1,0).

   Answer: 2p/3

   Solution: Points (1/2, $-\sqrt{3}$ /2) and (1/2, $\sqrt{3}$ /2) are 1 away from (1,0), so the points closer than 1 away on the circle are those with x > 1/2. The 120° sector of the circle has length (1/3)2pr = 2p/3.

3. How many sides does a polygon have if each of its internal angles has measure 130° or 173° ?

   Answer: 33

   Solution: The sum of the exterior angles in any polygon is 360°. If this polygon has m many angles of measure 130° and n many of measure 173°, then 50m+7n = 360. The only whole numbers m and n satisfying that equation are m = 3 and n = 30, so there are m+n = 33 interior angles, so there are also 33 sides.

4. Find the number c such that the semicircle $y = \sqrt{1-x^2}$ has equally much of its length above y = c as below.

   Answer: $\dfrac{\sqrt{2}}{2}$, or $c = \dfrac{\sqrt{2}}{2}$

   Solution: Starting at (1,0) and proceeding counterclockwise one-fourth of the length around the semicircle we arrive at $(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2})$, with a second quarter of the length lying beyond $(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2})$. So, half lies above and half below $y = \dfrac{\sqrt{2}}{2}$.

5. A right circular cone has slant height three times its height. Find the ratio formed by dividing the cube of its circumference by its volume.

Answer: $48\sqrt{2}\ p^2$

Solution: If its height is h and slant height has length 3h, then (from a side view of the cone) by the Pythagorean theorem, its radius is $2\sqrt{2}$ h. So,

$$\frac{C^3}{V} = \frac{((2p)2\sqrt{2}h)^3}{(1/3)p(2\sqrt{2}h)^2h} = \frac{128\sqrt{2}p^3}{(8/3)p} = 48\sqrt{2}\ p^2.$$

6. Let lcm(a,b) denote the least common multiple of a and b. For fractions $\frac{p}{q}$ and $\frac{r}{s}$ in reduced form for positive integers p, q, r and s, let $\frac{p}{q} * \frac{r}{s}$ denote the result of reducing to lowest terms the fraction $\frac{\text{lcm}(p,s)}{\text{lcm}(q,r)}$.

Find $\frac{7}{18} * (\frac{8}{15} * \frac{9}{20})$

Answer: $\frac{7}{8}$

Solution: To find $\frac{8}{15} * \frac{9}{20}$ we reduce $\frac{\text{lcm}(8,20)}{\text{lcm}(15,9)} = \frac{40}{45}$ to get $\frac{8}{9}$. So to find $\frac{7}{18} * (\frac{8}{15} * \frac{9}{20})$, i.e., $\frac{7}{18} * \frac{8}{9}$, we reduce $\frac{\text{lcm}(7,9)}{\text{lcm}(18,8)} = \frac{63}{72}$ to get $\frac{7}{8}$.

7. Find the last digit of the expansion of 1234567^{2006}.

Answer: 9

Solution: The last digit is the same as for 7^{2006}, since the other digits of 1234567 do not affect the last digit of the expansion. The sequence formed by the last digits of 7^1, 7^2, 7^3, 7^4, 7^5, is 7, 9, 3, 1, 7, 9, 3, 1, ... So, since 2006 has remainder 2 upon division by 4, the answer is 9

. 8 . Let d denote the greatest common divisor of 2006^{765} and 765^{2006}. How many different positive divisors does d have?

Answer: 766

Solution: 2006 factors as $2 \times 17 \times 59$ and 765 factors as $5 \times 3^2 \times 17$, so $d = 17^{765}$. Its positive divisors are 17^0, 17^1, 17^2, ..., 17^{765}, of which there are 766 many divisors.

9. Which one of the following is *impossible* for the graph of a curve $y = f(x)$ with a horizontal asymptote on the right side of the graph? Just write the letter of the response.
(a) There is a highest point on $y = f(x)$, yet $f(x)$ increases over $0 < x < 8$.
(b) $f(x)$ crosses the x-axis infinitely many times over $0 < x < 8$.
(c) f is an *odd* function, but has no horizontal asymptote on the left side of its graph.
(d) $f(x) = x^p$ for some constant p.

Answer: (c)

Solution: (a) is possible, as with $y = -\frac{x+1}{x^2}$, with highest point (-2, 1/4).

(b) is possible, as with $y = \frac{\sin x}{x}$. (c) is *impossible*, since if the graph has 180° rotational symmetry, then an asymptote on the right side forces one to exist on the left side. (d) is possible, as with $y = x^{-1}$.

10. Find a formula for a 4th degree polynomial function f(x) which is <u>even</u> and has roots at x = -1 and x = 2, such that f(0) = 8. Express your answer in expanded form, not factored form.

Answer: $f(x) = 2x^4 - 10x^2 + 8$, or simply $2x^4 - 10x^2 + 8$

Solution: Since -1, 2 are roots and f is even, +1 and –2 are also roots, so f has factored form $f(x) = c(x-1)(x+1)(x-2)(x+2)$ for some constant c. Since f(0) = 8, we have that c = 2, and upon expanding we get
$f(x) = 2(x^2-1)(x^2-4) = 2x^4-10x^2+8$.

11. A parallelogram has sides of lengths 3 and 5 and has a 60° angle. Find the length of its longer diagonal.

Answer: 7

Solution: By the Law of Cosines, if c is the answer, then
$c^2 = a^2 + b^2 - 2ab \cos 120° = 9 + 25 - 30 (-1/2) = 49$, so c = 7.

12. There are constants a and b for which $\cos(3\theta) = (\cos\theta)(a + b \sin^2\theta)$ is an identity. Find b.

Answer: -4, or b = -4

Solution: At $\theta = 0$ we have that 1 = a. At θ = p/3 or 60° we have that $-1 = (1/2)(1+\frac{3b}{4})$, so b = -4. While this is a good way to find the only POSSIBLE values for a and b, for checking that this problem is correctly written we must check that $\cos(3\theta) = (\cos\theta)(1 - 4 \sin^2\theta)$ is indeed an identity. To see this, use the identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$
to learn that $\cos(3\theta) = \cos(2\theta) \cos\theta - \sin(2\theta) \sin\theta$
$= (1-2\sin^2\theta) \cos\theta - 2 \sin\theta \cos\theta \sin\theta = \cos\theta - 4 \cos\theta \sin^2\theta$
$= (\cos\theta)(1 - 4 \sin^2\theta)$. Students could use this method to find the answer in the first place.

13. A point P is known to lie on a line segment of length 2 that is tangent to the circle $x^2 + y^2 = 1$. Find the area comprised by the set of points in the plane that are possible locations for P.

Answer: 4p, or 4p sq. units

Solution: If the point of tangency is (0,1), then the possible locations for P are the points on the line segment joining (-2,1) and (2,1). Rotating the line segment along with the circle to account for other possible points of tangency, the segment sweeps out the region in question, which is
$\{(x,y) : 1 = x^2 + y^2 = 5 \}$. Its area is $p(\sqrt{5}^{\,2}) - p(1^2) = 4p$.

14. A fair coin is tossed repeatedly, stopping after the first time it lands heads side up. Find the probability that the number of times the coin gets tossed is odd.

Answer: $\frac{2}{3}$

Solution: Let p denote the answer. The probability that heads appears on the first flip equals 1/2. When the first toss is tails, which happens half the time, the probability that the number of flips remaining is even equals 1–p. Thus p satisfies the equation $p = \frac{1}{2} + \frac{1}{2}(1-p)$, so $p = \frac{2}{3}$. For another solution, we add up the separate probabilities that the number of flips is 1 or 3 or 5 or..., getting the geometric series $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \ldots$, whose total is $\frac{a}{1-r} = \frac{1/2}{1-1/4} = \frac{2}{3}$.

15. For $f(x) = \frac{1}{x-1}$, find the exact value of the positive number c for which $f(f(f(c))) = c$.

Answer: $\frac{1+\sqrt{5}}{2}$, or $\frac{1}{2} + \frac{\sqrt{5}}{2}$

Solution: $f(f(x)) = \dfrac{1}{\frac{1}{x-1} - 1} = \dfrac{x-1}{1-(x-1)} = \dfrac{x-1}{2-x}$, so $f(f(f(x))) = \dfrac{1}{\frac{x-1}{2-x} - 1}$ $= \dfrac{2-x}{x-1-(2-x)} = \dfrac{2-x}{2x-3}$. Solving $\frac{2-c}{2c-3} = c$, we get $c = \frac{1 \pm \sqrt{5}}{2}$. Discarding the negative solution because of the instructions, the answer is $\frac{1+\sqrt{5}}{2}$.

Another way is to make the "lucky" guess that there is a positive number c for which $f(c) = c$, since for such a number certainly $f(f(f(c))) = c$. Solving $x = \frac{1}{x-1}$, or $x^2 - x = 1$, we get the same answer.

16. Which one of the following functions has exactly the same graph as $y = |x|$? Just give the correct letter name f, g, h or i as your answer.

$f(x) = e^{(1/2)\ln(x^2)}$ $\qquad g(x) = \sqrt{\ln(e^{x^2})}$ $\qquad h(x) = \sin(\sin^{-1}|x|)$ $\qquad i(x) = \dfrac{|x^2+2x|}{|x+2|}$

Answer: g

Solution: For f, although $\ln(x^2)$ does equal $2 \ln|x|$, there is still the trouble that f(0) is undefined, not 0, so the graph of f is not quite the same as for $y = |x|$. For g, $\ln(e^{x^2})$ does simplify to x^2, whose square roots equals $|x|$, so this function has exactly the same graph. For h, if $x > 1$ or $x < -1$, h(x) is undefined, so the graph of h is not quite as desired, even though for x between -1 and 1 function h matches $|x|$. Since i(-2) is undefined, i is not quite as desired either, although i matches $|x|$ at all other inputs. A graphing calculator is not much help on this problem, although h can be eliminated easily that way.

17. Consider the tangent line to $y = \sec^{-1}(\sqrt{x}\,)$ at $x = 2$. Find the x-coordinate of the point where that line crosses the x-axis. (The derivative of $\sec^{-1}x$ is $\dfrac{1}{x\sqrt{x^2-1}}$ when $x = 1$.)

Answer: $\sqrt{2} - p$, or $x = \sqrt{2} - p$

Solution: By the chain rule, $y' = \frac{1}{\sqrt{x}\sqrt{\sqrt{x}^{2}-1}}\frac{1}{2}\ x^{-1/2}$ at x = 2, which simplifies nicely to 1/4. So, since y = p/4 when x =2, in point-slope form the tangent line has equation $y - \frac{p}{4} = \frac{1}{4}(x-2)$. For the x-intercept, setting y = 0 we get $-\frac{p}{4} = \frac{1}{4}(x-2)$, so -p = x –2, so x =2– p.

18. For a, b > 1, the area of the elliptical region $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is pab, simply because the ellipse is the result of stretching circular region $x^2 + y^2 = 1$
until it is "a times as wide" and "b times as tall" as before. How much area in $\frac{x^2}{16} + \frac{y^2}{9} = 1$ lies to the right of x = 2 ?

Answer: $4p - 3\sqrt{3}$

Solution: Start with the region R in $x^2 + y^2 = 1$ lying to the right of x = 1/2. Stretching it 4 times as wide and 3 times as tall with respect to the x and y axes, we get the desired region, so we just multiply the area of R by 12 to get the answer. For the area of R, notice that R lies in a sector of $x^2 + y^2 = 1$ bounded by radii from the center to $(1/2, -\sqrt{3}/2)$ and $(1/2, \sqrt{3}/2)$. That's a 120° sector, with area 1/3 of a circle, so area p/3. However, the part of R not in that sector is a triangle which, on its side, has base length $\sqrt{3}$ and height 1/2, so has area $\sqrt{3}/4$. Thus the answer is $12(p/3 - \sqrt{3}/4) = 4p - 3\sqrt{3}$.

19. How many positive integers have no prime factors larger than 12, and have no divisors larger than 1 that are perfect squares?

Answer: 32

Solution: Such a number has prime factorization $2^a 3^b 5^c 7^d 11^e$ where each of a, b, c, d, e is 0 or 1 (since if a prime p appears raised to a power 2 or more then the integer is divisible by the square p^2). There are 2 choices for each of the 5 exponents, so there are 2^5 such numbers.

20. For the triangle with vertices at (0,0), (0,3) and (2,1), find the number c for which the line x = c divides the triangle into pieces having equal areas.

Answer: $2 - \sqrt{2}$, or $c = 2 - \sqrt{2}$

Solution: The triangle on its side has base length 3 and height 2, so has area 3. For the triangular portion of it to the right of x = c, also viewed on its side, let b denote its base length and h its height. Then by similar triangles, $\frac{b}{3} = \frac{h}{2}$. Also, $\frac{bh}{2} = \frac{3}{2}$, since this portion has half the full area. Substituting $b = \frac{3h}{2}$ into this latter equation and solving for h, we have
$h = \sqrt{2}$. So, $c = 2 - \sqrt{2}$. Yes, students can use calculus, but this or some other geometric approach is probably easier.