OHMIO 2006 Ciphering[1]

Practice Problem:
The sum of all but one of the interior angles of a convex polygon equals $2570°$. The remaining angle contains how many degrees?

Answer: 130 or $130°$

Solution: The sum of the angles equals
$(n-2)180$. Therefore, if x is the unknown angle, $(n-2)180 = 2570° + x$, with $0° < x < 180°$.
If n=17, then $(n-2)180 = 15(180) = 2700°$, and $x = 130°$

Question 1

Find the following sum:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{99\cdot 100}$$

Answer: $\frac{99}{100}$

Solution: Look at the pattern

$$\frac{1}{1\cdot 2}=\frac{1}{2},\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3},\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{2}{3}+\frac{1}{12}=\frac{3}{4},\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{3}{4}+\frac{1}{20}=\frac{4}{5},$$

Therefore $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{99\cdot 100}=\frac{99}{100}$, or breaking them into partial fractions, you have

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\ldots+\frac{1}{99}-\frac{1}{100}=1-\frac{1}{100}=\frac{99}{100}$$

Question 2

Let A be the 34th term in the arithmetic sequence 1, 4, 7, ...

Let B be the sum of the first 21 terms of the arithmetic sequence 2, 6, 10, ...

Let C be the ninth term in the geometric sequence 1, $\sqrt{2}$, 2, ...

Let D be the sum of the infinite geometric series $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\ \ldots$

Find $A+B+C+D$

Answer: 1000
Solution:

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$A = t_{34} = 1 + 33(3) = 100$ $B = S_{21} = \frac{21}{2}(2 + 2 + 20(4)) = 882$ $C = t_9 = 1(\sqrt{2})^8 = 16$

$D = \dfrac{\frac{2}{3}}{1 - \frac{2}{3}} = 2$ $A + B + C + D = 100 + 882 + 16 + 2 = 1000$

Question 3

The ellipse $4x^2 + 9y^2 - 16x - 18y - 11 = 0$ Has two vertices in the first quadrant. Find, in slope-intercept form, the equation of the line containing these two vertices.

Answer: $y = -\frac{2}{3}x + \frac{13}{3}$

Solution: In standard form the ellipse is $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$, which has a center at $(2,1)$

And first quadrant vertices at $(2,3)$ and $(5,1)$, m=$-\frac{2}{3}$ and $y = -\frac{2}{3}x + \frac{13}{3}$

Question 4

What is the value of $\sin\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)$?

Answer: $\frac{\sqrt{6}}{2}$

Solution:

Square to get $\left(\sin\frac{\pi}{12} + \cos\frac{\pi}{12}\right)^2 = 1 + 2\sin\frac{\pi}{12}\cos\frac{\pi}{12} = 1 + \sin\frac{\pi}{6} = 1 + \frac{1}{2} = \frac{3}{2}$,

$\sin\frac{\pi}{12} + \cos\frac{\pi}{12} > 0$, so $\sin\frac{\pi}{12} + \cos\frac{\pi}{12} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

Question 5

What is the units digit of the sum $5^{401} + 12^{401} + 13^{401}$?

Answer: 0

Solution:

5^n always has a units digit of 5

12^n cycles 2,4,8,6,2,4,8,6,... so 12^{401} has the same units digit as 12^1

13^n cycles 3,9,7,1,3,9,7,1,... so 13^{401} has the same units digit as 13^1

and $5 + 2 + 3$ has a units digit of 0

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Question 6

You are walking across a railroad trestle bridge when you hear the sound of an approaching train. The bridge is one mile across, and you are $\frac{3}{4}$ of the way across, and heading for the oncoming train. You know the following:

> You can just make it off the bridge in time, no matter which way you run, as long as you run at your top speed.
> The train is traveling at 30 mph.

What is the top speed at which you can run?

Answer: 15 mph

Solution: Let t represent the time necessary for the train to reach the trestle.
If she runs away from the train, she has to run $\frac{3}{4}$ of a mile and the train has to travel t hours to the trestle and the one mile across. If she runs toward the train, she travels $\frac{1}{4}$ mile and the train travels for t hours.

$\frac{\frac{3}{4}}{x} = t + \frac{1}{30}$, $\frac{\frac{1}{4}}{\text{x}} = t$, simplifying $\frac{3}{4x} = t + \frac{1}{30}$ and $\frac{1}{4x} = t$, $\frac{2}{4x} = \frac{1}{30}$, $x = 15\text{mph}$

Question 7

343 unit cubes are painted blue and stacked into one large cube. The surface of the large cube is painted orange. How many cubes are formed that are not painted orange?
Answer: 225
Solution:
How many solely blue cubes?
1: 5*5*5
8: 4*4*4 1+8+27+64+125=225
27: 3*3*3
64: 2*2*2
125: unit cubes

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Question 8

Find the volume of a solid whose base is bounded by the circle $x^2 + y^2 = 4$ and cross sections taken perpendicular to the x-axis are squares.

Answer: $\frac{128}{3}$

Solution:

The sides of the square are $2y = 2\sqrt{4-x^2}$ Area of squares = side squared = $\left(2\sqrt{4-x^2}\right)^2$

Volume = $\int_{-2}^{2} 4\left(4-x^2\right)dx = 4\left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \frac{128}{3}$

TB 1

Let $f(x) = 2x^3 + 3x^2 - 8x + 12$.
Let R be the sum of the reciprocals of the roots of f.
Let S be the sum of the squares of the roots of f.
What is $3R + 4S$?

Answer: 43

Solution:
$x^3 + \frac{3}{2}x^2 - 4x + 6 = 0$ gives us the equations involving the roots $a,\ b,$ and c

$a + b + c = -\frac{3}{2}$

$ab + bc + ac = -4$

$abc = -6$

$R = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc + ac + ab}{abc} = \frac{-4}{-6} = \frac{2}{3}$

$(\text{a+b+c})^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ or

$\text{S}=(\text{a+b+c})^2 - 2ab - 2ac - 2bc =$

$\text{S}=\left(-\frac{3}{2}\right)^2 - 2(-4) = \frac{9}{4} + 8 = \frac{41}{4}$

$3\left(\frac{2}{3}\right) + 4\left(\frac{41}{4}\right) = 43$

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TB 2

$$\lim_{h\to 0}\frac{\left(\sin\left(\frac{\pi}{3}+h\right)\right)^3-\left(\frac{3\sqrt{3}}{8}\right)}{h}=$$

Answer: $\frac{9}{8}$

Solution:

$$\left.\frac{d(\sin x)^3}{dx}\right|_{x=\frac{\pi}{3}}=3(\sin x)^2\cos x\Big|_{x=\frac{\pi}{3}}=3\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)=3\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)=\frac{9}{8}$$